

A Clifford algebraic framework for Coxeter group theoretic computations

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Abstract Real physical systems with reflective and rotational symmetries such as viruses, fullerenes and quasicrystals have recently been modeled successfully in terms of three-dimensional (affine) Coxeter groups. Motivated by this progress, we explore here the benefits of performing the relevant computations in a Geometric Algebra framework, which is particularly suited to describing reflections. Starting from the Coxeter generators of the reflections, we describe how the relevant chiral (rotational), full (Coxeter) and binary polyhedral groups can be easily generated and treated in a unified way in a versor formalism. In particular, this yields a simple construction of the binary polyhedral groups as discrete spinor groups. These in turn are known to generate Lie and Coxeter groups in dimension four, notably the exceptional groups D_4 , F_4 and H_4 . A Clifford algebra approach thus reveals an unexpected connection between Coxeter groups of ranks 3 and 4. We finally discuss how to extend these considerations and computations to the Conformal Geometric Algebra setup.

IPPP/12/49, DCPT/12/98

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1 Introduction

Physical systems have to obey the mathematical laws of geometry; in particular, if they possess symmetry – such as invariance under reflections and rotations – this symmetry is heavily constrained by purely geometric considerations. For instance, many physical systems in biology (viruses), chemistry (fullerenes) and physics (crystals and quasicrystals) have polyhedral symmetries. These polyhedral symmetry groups are generated by reflections; via the Cartan-Dieudonné theorem an even number of reflections amounts to a rotation, and physical systems may be invariant only under this rotational (chiral) part, or the full reflection group.

Coxeter group theory [5, 22] axiomatises reflections from an abstract mathematical point of view. Coxeter groups thus encompass the finite Euclidean reflection groups, which include the symmetry groups of the Platonic solids as well as the Weyl groups of simple Lie algebras. A subset of these groups are non-crystallographic, i.e. they describe symmetries that are not compatible with lattices in dimensions equal to their rank. They include the groups H_2 , H_3 and the largest non-crystallographic group H_4 , which are the only Coxeter groups generating rotational symmetries of order 5. The full icosahedral group H_3 and its rotational subgroup I are of particular practical importance, as H_3 is the largest discrete symmetry group of physical space. Thus, many 3-dimensional systems with ‘maximal symmetry’, like viruses in biology [43, 4, 45, 24, 46], fullerenes in chemistry [31, 30, 44, 32] and quasicrystals in physics [25, 40, 37, 35, 42], can be modeled using Coxeter groups.

Clifford’s Geometric Algebra [16, 15] is a complementary framework that focuses on the geometry of the physical space(-time) that we live in and its given Euclidean/Lorentzian metric. This exposes more clearly the geometric nature of many problems in mathematics and physics. In particular, Clifford’s Geometric Algebra has a uniquely simple formula for performing reflections. Previous research appears to have made exclusive use of one framework at the expense of the other. Here, we combine both paradigms, which results in geometric insights from Geometric Algebra that apparently have been overlooked in Coxeter theory thus far. This approach also has computational and conceptual advantages over standard techniques, in particular through a spinorial or conformal point of view. Hestenes [18] has given a thorough treatment of point and space groups in Geometric Algebra, and Hestenes and Holt [19] have discussed the crystallographic point and space groups from a conformal point of view. Here, we are interested in applying Geometric Algebra in the Coxeter framework, in particular in the context of root systems, non-crystallographic groups and quasicrystals.

This paper is organised as follows. Section 2 introduces how systems are currently modeled in terms of Coxeter groups, and what kind of computations arise in this context. In Section 3, we present a versor formalism in which the full, chiral and binary polyhedral groups can all be easily generated and treated within the same framework. In particular, this yields a construction of the binary polyhedral groups, which we will discuss further in Section 4. In Section 5, we briefly outline

how to extend this treatment to the conformal setup. We conclude with a summary and possible further work in Section 6.

2 Coxeter formulation

Coxeter groups are abstract groups describable in terms of mirror symmetries [5]. The elements of finite Coxeter groups can be visualised as reflections at planes through the origin in a Euclidean vector space V . In particular, for $v, \alpha \in V$, then

$$v \rightarrow r_\alpha v = v' = v - \frac{2\alpha \cdot v}{\alpha \cdot \alpha} \alpha \quad (1)$$

corresponds to a Euclidean reflection r_α of the vector v at a hyperplane perpendicular to the ‘root’ vector α .

Finite Coxeter groups describe the properties of physical structures, e.g. of a viral protein container or a carbon onion, at a given radial level. In order to obtain information on how structural properties at different radial levels are collectively constrained by symmetry, affine extensions of these groups need to be considered. Affine extensions are constructed in the Coxeter framework by adding affine reflection planes not containing the origin [36]. A detailed account of this construction is presented elsewhere [38, 10, 11], but essentially the affine extension amounts to making the reflection group G non-compact by adding a translation operator T . The structures of viruses follow several different extensions of the icosahedral group I by translation operators [26, 12]. Thus, a wide range of empirical observations in virology can be explained by affine Coxeter groups. We now discuss 2D counterparts to the 3D point arrays that predict the architecture of viruses and fullerenes, and explain in what sense the translation operators are distinguished.

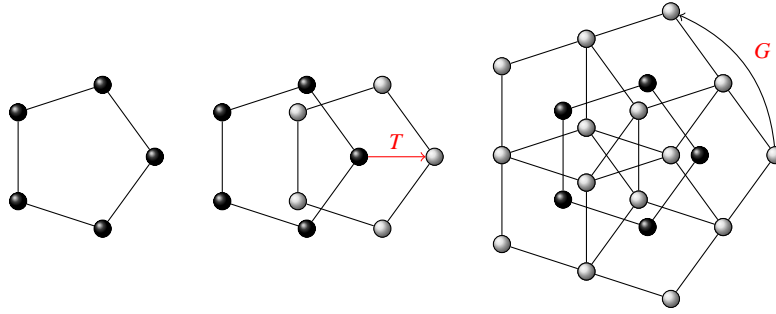


Fig. 1 The action of an affine Coxeter group on a pentagon. The translation operator T generates extended point arrays, whilst the compact part G makes the resulting point set rotationally symmetric. Blueprints with degeneracies due to coinciding points correspond to non-trivial group structures and can be used in the modeling of viruses.

For illustration purposes, let us consider a similar construction for a pentagon of unit size, as shown in Fig. 1. The non-compact translation operator T , here chosen to also be of unit length, creates a displaced version of the pentagon. The action of the symmetry group G of the pentagon then generates further copies in such a way that the final point array displays the same rotational symmetries.

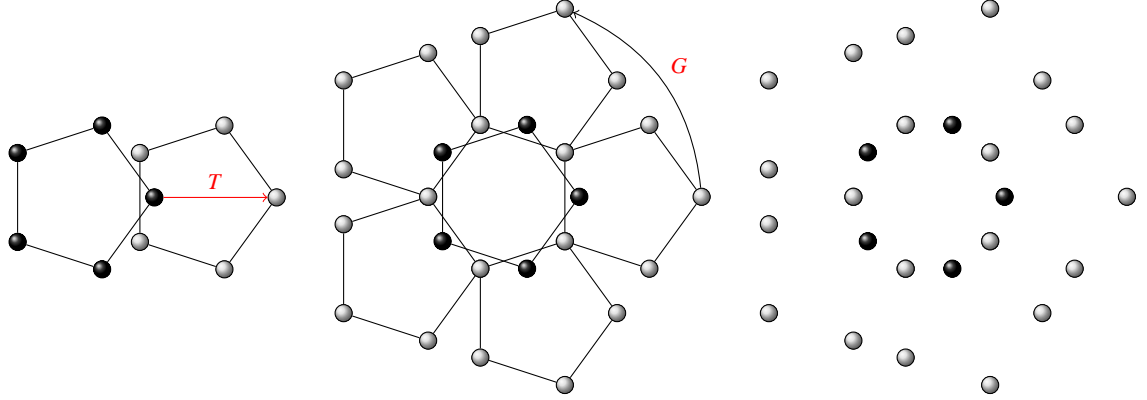


Fig. 2 Translation by the golden ratio results in a point set whose constituent polygons are simultaneously constrained by the affine symmetry.

The translation operator we have chosen for this example is distinguished because several of the generated points lie on more than one pentagon, for instance the innermost points, or the midpoints of the edges of the large outer pentagon. Certain distinguished translations lead to such point sets with degenerate points, which therefore have lower cardinality than those obtained by a random translation (here 15 points as opposed to 25). This degeneracy yields a non-trivial mathematical structure at the group level, and the corresponding blueprints in three dimensions can be used to model icosahedral viruses.

Fig. 2 shows a similar example for a translation of length of the golden ratio $\tau = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$. The resulting point set also has degenerate cardinality (20), and consists of an inner decagon and an outer pentagon. Affine symmetry here means that the relative sizes of the decagon and pentagon are fixed by the group structure. This is a powerful geometric tool for constraining real systems.

The computations necessary in this context are therefore translations, reflections and rotations; one also needs to be able to check degeneracy of points. In the usual vector space approach, these operations are implemented via matrices. We instead develop here a versor implementation. This has some computational advantages, as well as offering surprising geometric insights, as we shall see later. Whilst the computational complexity for 3-dimensional applications is limited, equivalent computations in four dimensions, where H_4 has order 14,400 and H_4 -symmetric polytopes have upwards of 120 and 600 vertices, are rather more complex.

In the Coxeter setting, therefore, the reflections are fundamental; Geometric Algebra is very efficient at encoding reflections algebraically, and at performing computations with clear geometric content. However, the two frameworks do not appear to have been combined previously. We therefore explore which benefits a Clifford algebraic description might offer for Coxeter group theoretic considerations.

3 Versor framework

The geometric product $xy = x \cdot y + x \wedge y$ of Geometric Algebra [16, 20, 17, 15] provides a very compact and efficient way of handling reflections in any number of dimensions, and thus by the Cartan-Dieudonné theorem also rotations. For a unit vector α , the two terms in the formula for a reflection of a vector v in the hyperplane orthogonal to α from Eq. (1) simplify to the double-sided action of α via the geometric product

$$v \rightarrow v' = -\alpha v \alpha. \quad (2)$$

This prescription for reflecting vectors in hyperplanes is remarkably compact, and applies more generally to all multivectors. Even more importantly, from the Cartan-Dieudonné theorem, rotations are the product of successive reflections. For instance, compounding the reflections in the hyperplanes defined by the unit vectors α_i and α_j results in a rotation in the plane defined by $\alpha_i \wedge \alpha_j$

$$v'' = \alpha_j \alpha_i v \alpha_i \alpha_j =: \tilde{R} v R, \quad (3)$$

where we have defined the rotor $R = \alpha_i \alpha_j$ and the tilde denotes the reversal of the order of the constituent vectors $\tilde{R} = \alpha_j \alpha_i$. Rotors satisfy $\tilde{R} R = R \tilde{R} = 1$ and themselves transform single-sidedly under further rotations. They thus form a multiplicative group under the geometric product, called the rotor group, which is essentially the Spin group, and thus a double-cover of the special orthogonal group. Objects in Geometric Algebra that transform single-sidedly are called spinors, so that rotors are normalised spinors.

In fact, the above two cases are examples of a more general theorem on the Geometric Algebra representation of orthogonal transformations. In analogy to the vectors and rotors above, a versor is a multivector $A = a_1 a_2 \dots a_k$ which is the product of k non-null vectors a_i ($a_i^2 \neq 0$). These versors also form a multiplicative group under the geometric product, called the versor group. The Versor Theorem [17] then states that every orthogonal transformation \underline{A} of a vector v can be expressed via unit versors in the canonical form

$$\underline{A} : v \rightarrow v' = \underline{A}(v) = \pm \tilde{A} v A, \quad (4)$$

where the \pm -sign defines its parity. Unit versors are double-valued representations of the respective orthogonal transformation. Even versors form a double covering of the special orthogonal group, called the Spin group. The versor realisation of the

orthogonal group is much simpler than conventional matrix approaches, in particular in the Conformal Geometric Algebra setup, where one uses the fact that the conformal group $C(p, q)$ is homomorphic to $O(p+1, q+1)$ to treat translations as well as rotations in a unified versor framework (see Section 5).

We now consider which benefits such a versor approach can offer for Coxeter computations (more details are contained in [7, 8]). We begin with the simple roots (vertex vectors) which completely characterise a given Coxeter group, and consider their closure under mutual reflections (the root system). We then compute the rotors derivable from all these root vectors, which encode the rotational part of the respective polyhedral group via the double-sided action in Eq. (3). The rotor group defined by single-sided action can in fact be shown to realise the respective binary polyhedral group. Finally, including the versors of the form $\alpha_i \alpha_j \alpha_k$ gives a realisation of the full polyhedral group (the Coxeter group).

Theorem 3.1 (Reflections/Coxeter groups and polyhedra/root systems) *Take the three simple roots for the Coxeter groups $A_1 \times A_1 \times A_1$ (respectively $A_3/B_3/H_3$). Geometric Algebra reflections in the hyperplanes orthogonal to these vectors via Eq. (2) generate further vectors pointing to the 6 (resp. 12/18/30) vertices of an octahedron (resp. cuboctahedron/cuboctahedron with an octahedron/icosidodecahedron), giving the full root system of the group.*

For instance, the simple roots for $A_1 \times A_1 \times A_1$ are $\alpha_1 = e_1$, $\alpha_2 = e_2$ and $\alpha_3 = e_3$ for orthonormal basis vectors e_i . Reflections amongst those then also generate $-e_1$, $-e_2$ and $-e_3$, which all together point to the vertices of an octahedron.

By the Cartan-Dieudonné theorem, combining two reflections yields a rotation, and Eq. (3) gives a rotor realisation of these rotations in Geometric Algebra.

Theorem 3.2 (Spinors from reflections) *The 6 (resp. 12/18/30) reflections in the Coxeter group $A_1 \times A_1 \times A_1$ (resp. $A_3/B_3/H_3$) generate 8 (resp. 24/48/120) rotors.*

For the $A_1 \times A_1 \times A_1$ example above, the spinors thus generated are ± 1 , $\pm e_1 e_2$, $\pm e_2 e_3$ and $\pm e_3 e_1$. In fact, these groups of discrete spinors yield a novel construction of the binary polyhedral groups.

Theorem 3.3 (Spinor groups and binary polyhedral groups) *The discrete spinor group in Theorem 3.2 is isomorphic to the quaternion group Q (resp. binary tetrahedral group $2T$ /binary octahedral group $2O$ /binary icosahedral group $2I$).*

The isometry group of space is $O(3)$, of which the full polyhedral (Coxeter) groups are discrete subgroups. However, $O(3)$ is globally $SO(3) \times \mathbb{Z}_2$, where $SO(3)$ is the subgroup of pure rotations (or the chiral part). $SO(3)$ is still not simply-connected, but is doubly covered by the Spin group $\text{Spin}(3) \simeq SU(2)$ (in fact, it is $SO(3) \times \mathbb{Z}_2$ locally, i.e. a fibre bundle). Thus, the chiral polyhedral groups are discrete subgroups of $SO(3)$, the full polyhedral groups (Coxeter) are their preimage in $O(3)$, and the binary polyhedral groups are their preimage under the universal covering in $\text{Spin}(3)$.

Through the versor theorem, we can therefore describe all three types of groups in the same framework. Vectors are grade 1 versors, and rotors are grade 2 versors.

Table 1 Versor framework for a unified treatment of the chiral, full and binary polyhedral groups.

Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full)	$x \rightarrow \pm \tilde{A}xA$
$Spin(3)$	binary	spinors R under spinor multiplication $(R_1, R_2) \rightarrow R_1 R_2$

For instance, the 60 rotations of the chiral icosahedral group are given by 120 rotors acting as $\alpha_i \alpha_j v \alpha_j \alpha_i$. 60 operations of odd parity are defined by 120 grade 1 and grade 3 versors (with vector and trivector parts) acting as $-\alpha_i \alpha_j \alpha_k v \alpha_k \alpha_j \alpha_i$. However, 30 of them are just the 15 true reflections given by pure vectors, leaving another 45 rotoinversions. Thus, the Coxeter group, the full icosahedral group $H_3 \subset O(3)$ is expressed in accordance with the versor theorem. Alternatively, one can think of 60 rotations and 60 rotoinversions, making $H_3 = I_h = I \times \mathbb{Z}_2$ manifest. However, the rotations operate double-sidedly on a vector, such that the versor formalism actually provides a 2-valued representation of the rotation group $SO(3)$, since the rotors R and $-R$ encode the same rotation. Since $Spin(3)$ is the universal 2-cover of $SO(3)$, the rotors form a realisation of the preimage of the chiral icosahedral group, i.e. the binary icosahedral group. Thus, in the versor approach, we can treat all these different groups in a unified framework, whilst maintaining a clear conceptual separation. In Table 1, we summarise how the three different types of polyhedral groups are realised in the versor framework.

4 Construction of the binary polyhedral groups

In this section, we consider further the implications of our construction of the binary polyhedral groups. Since Clifford algebra is well known to provide a simple construction of the Spin groups, it is perhaps not surprising – from a Clifford algebra point of view – to find that the discrete rotor groups realise the binary polyhedral groups. However, this construction does not seem to be known, and from a Coxeter group point of view, it leads to rather surprising consequences.

The Geometric Algebra construction of the binary polyhedral groups is via rotors with (single-sided) rotor multiplication. It is then straightforward to check the group axioms, multiplication table, conjugacy classes and the representation theory. However, it is also known that the binary polyhedral groups generate some Coxeter groups of rank 4, in particular Q , $2T$, $2O$ and $2I$ generate $A_1 \times A_1 \times A_1 \times A_1$, D_4 , F_4 and H_4 , respectively, as summarised in Table 2. From a Coxeter perspective, this is surprising. However, in Geometric Algebra, spinors have a natural 4-dimensional Euclidean structure given by $\psi\tilde{\psi}$, and can thus also be interpreted as vectors in a 4D Euclidean space. In fact, one can show that these vertex vectors are again root systems [9, 8, 22], which generate the respective rank-4 Coxeter groups. This demonstrates how in fact the rank-4 groups can be derived from the rank-3 groups

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$		Q	$A_1 \times A_1 \times A_1 \times A_1$	
A_3		$2T$	D_4	
B_3		$2O$	F_4	
H_3		$2I$	H_4	

Table 2 Correspondence between the rank-3 and rank-4 Coxeter groups. The spinors generated from the reflections contained in the respective rank-3 Coxeter group via the geometric product are realisations of the binary polyhedral groups Q , $2T$, $2O$ and $2I$, which in turn generate (mostly exceptional) rank-4 groups.

via the geometric product of Clifford's Geometric Algebra. This connection has so far been overlooked in Coxeter theory. This 'induction' of higher-dimensional root systems via spinors of lower-dimensional root systems is complementary to the well-known top-down approaches of projection (for instance from E_8 to H_4 [41, 37, 29, 28, 11]), or of taking subgroups by deleting nodes in Coxeter-Dynkin diagrams. It is particularly interesting that this inductive construction relates the exceptional low-dimensional Coxeter groups H_3 , D_4 , F_4 and H_4 to each other as well as to the series A_n , B_n and D_n in novel ways. In particular, it is remarkable that the exceptional dimension-four phenomena D_4 (triality), F_4 (the largest crystallographic Coxeter group in 4D) and H_4 (the largest non-crystallographic Coxeter group) are seen to arise from three-dimensional geometric considerations alone, and it is possible that their existence is due to the 'accidentalness' of the spinor construction. This spinorial view could thus open up novel applications in Coxeter and Lie group theory, as well as in polytopes (e.g. A_4), string theory and triality (D_4), lattice theory (F_4) and quasicrystals (H_4).

5 Conformal Geometric Algebra and Coxeter groups

The versor formalism is particularly powerful in the Conformal Geometric Algebra approach [20, 15, 6]. The conformal group $C(p, q)$ is 1 – 2-homomorphic to $O(p + 1, q + 1)$ [1, 2], for which one can easily construct the Clifford algebra and find rotor implementations of the conformal group action, including rotations and translations. Thus, translations can also be handled multiplicatively as rotors, for flat, spherical and hyperbolic space-times, simplifying considerably more traditional approaches and allowing novel geometric insight. Hestenes [18, 19, 21] has applied this framework to point and space groups, which is fruitful for the crystallographic groups, as lattice translations can be treated on the same footing as the rotations and reflections. An extension of the conformal framework to translations in the case of non-crystallographic Coxeter groups could have interesting consequences for quasilattice theory [25, 40], in particular for quasicrystals induced via projection from

higher dimensions (e.g. via the cut-and-project method) [37, 11, 23]. We therefore briefly outline the basics of such a construction.

Let us consider the space of signature $(+, +, +, +, -)$ achieved by adjoining two additional orthogonal unit vectors e and \bar{e} to the algebra of space [20]. It is therefore spanned by the unit vectors

$$e_1, e_2, e_3, e, \bar{e}, \text{ with } e_i^2 = 1, e^2 = 1, \bar{e}^2 = -1. \quad (5)$$

From these two unit vectors we can define the two null vectors

$$n \equiv e + \bar{e}, \quad \bar{n} \equiv e - \bar{e}. \quad (6)$$

One can then map a 3D vector x into the space of null vectors in the conformal space by defining

$$X \equiv F(x) := x^2 n + 2\lambda x - \lambda^2 \bar{n}. \quad (7)$$

X being null allows for a homogeneous (projective) representation of points, i.e. they are represented by a ray in the conformal space, which tends to be more numerically robust in applications. Here, λ is a fundamental length scale that is needed in order to make this expression dimensionally homogeneous, as we think of the position vector x as a dimensionful quantity [33]. The notation in terms of the Amsterdam protocol is $e = e_+$, $\bar{e} = e_-$, $n = n_\infty$ and $\bar{n} = n_0$. This notation is also consistent with the notion that the above mapping is essentially an embedding into the projective null cone of the embedding space. Originally due to Dirac [14], the idea is that the projective null cone inherits the $SO(4, 1)$ invariance of the ambient space in which $SO(4, 1)$ acts linearly, thereby endowing the projective null cone with a non-linearly realised conformal structure.

The vectors e and \bar{e} and therefore also n and \bar{n} are orthogonal to x and hence anti-commute with it, i.e. $-x^{-1}nx = n$ and $-x^{-1}\bar{n}x = \bar{n}$. Thus, the CGA implementation of a reflection $y' = -x^{-1}yx$ is given by

$$-x^{-1}F(y)x = F(y') = F(-x^{-1}yx). \quad (8)$$

Given the simple roots, one can again generate the whole root system via successive reflections as shown in Fig. 3 (left). We firstly notice that the conformal representation of a root vector $F(\alpha)$ is now different from the implementation of the reflection α encoded by it. These two roles were treated on an equal footing in 3D, and it is debatable whether the conceptual advantages of CGA outweigh this disadvantage.

Secondly, it is often argued that the implementation of rotations in CGA is given by $F(x') = RF(x)\bar{R}$, since R only contains even blades and thus commutes with the vectors n and \bar{n} such that $Rn\bar{R} = n$ and $R\bar{n}\bar{R} = \bar{n}$. However, via the Cartan-Dieudonné theorem, every rotation is given by an even number of successive reflections. Thus, it can be seen that the rotor transformation law actually follows from the more fundamental reflection law in Eq. (8). From the previous sections, we know that the spinors generated by the root vectors are important for the construction of the binary polyhedral groups and 4D polytopes. However, the 3D geometric product does

not straightforwardly extend to CGA, such that the spinors and other multivectors are not treated in the same way as vectors. The operators encoding the conformal rotations, however, are still given by the 3D rotors, so that little seems to be gained by going to the conformal setup from the spinorial point of view.

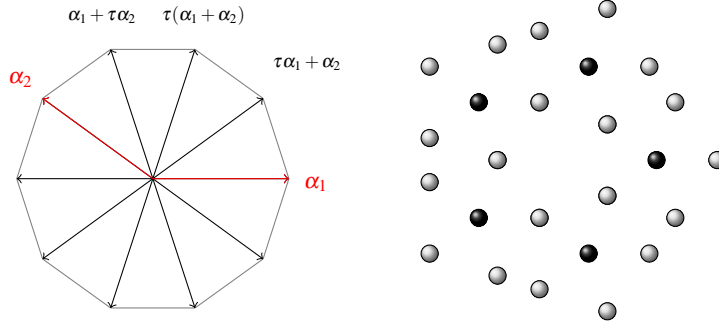


Fig. 3 In the conformal setup, reflections generated by the simple roots (here e.g. α_1 and α_2 for a simple two-dimensional example, H_2) according to Eq. (8) again generate, for instance, the H_2 non-crystallographic root system, the decagon (left). CGA rotor translations via Eq. (9) act multiplicatively, but yield quasicrystalline point sets consistent with the 3D approach; for instance, on the right we show the effect of a translation with length the inverse of the golden ratio acting on a pentagon, in analogy to Figs 1 and 2.

A very salient feature of Conformal Geometric Algebra is that a translation $x \rightarrow x + a$ by a vector a is given by a rotor

$$T_a = \exp\left(\frac{na}{2\lambda}\right) = 1 + \frac{na}{2\lambda}. \quad (9)$$

It is easily checked that this has the desired effect of $T_a F(x) \tilde{T}_a = F(x + a)$, and therefore does indeed represent a 3D translation as a rotor in Conformal Geometric Algebra. One can thus treat reflections, rotations and translations multiplicatively in a unified framework. This allows for a unified construction of the type of point arrays considered earlier, and indeed the construction is entirely equivalent to the lower-dimensional construction (as it must), and can be straightforwardly verified, for instance, for the non-crystallographic groups $I_2(n)$, H_3 and H_4 . In Fig. 3, we show examples of such a root system and quasicrystal-like point array derived entirely in the conformal setup, as a proof of principle. The root system shown is that of H_2 , and the point array is obtained via the action of a translation of length the inverse of the golden ratio on a pentagon.

The CGA approach is naturally more computationally intensive than the 3D approach; however, this could be compensated for by increased numerical stability, as the origin is simply represented by scalar multiples of \bar{n} . Treating both rotations and translations on the same footing as multiplicative rotors is also a nice conceptual shift. However, there are also drawbacks to the conformal approach. Firstly, the con-

formal representation of the root vectors is different from their action as generators of reflections. The relationship between these two functions was more transparent in the conventional approach. Secondly, the rotors encoding rotations are also the 3D spinors. Thus, CGA affords a nice representation of GA vectors, but not necessarily of the whole GA multivector structure.

Following [34], an interesting approach might be to work in a curved space, for which only one extra dimension is necessary (e or \bar{e}), which should simplify the computations somewhat. One may then finally take the zero curvature limit in order to recover the Euclidean space results. For instance, for spacetime, the conformal group $C(1, 3)$ is 15-dimensional. It has certain well-known ten-dimensional groups as stabiliser subgroups, i.e. groups of transformations that leave a given point (ray) y invariant. If y is spacelike, one gets an $SO(2, 3)$ stabiliser subgroup, i.e. the Anti de Sitter group. Likewise, for timelike y one obtains the de Sitter group $SO(1, 4)$ as the stabiliser (preferred choices for y are here e and \bar{e}). Lastly, when one chooses a null y (e.g. n), one gets an $ISO(1, 3)$ subgroup, which is just the Poincaré group [39, 3]. Thus, taking the zero curvature limit essentially corresponds at the group level to the Wigner-Inönü contraction that yields the Poincaré group from the de Sitter and Anti de Sitter groups.

6 Conclusions

We have investigated what insight a Geometric Algebra description, which lends itself to applications of reflections, can offer when applied to the Coxeter (reflection) group framework. The corresponding computations are conceptually revealing, both for applications to real systems and for purely mathematical considerations. The implementation of orthogonal transformations as versors rather than matrices offers some computational and conceptual advantages, in both the conventional and the conformal approaches. The main benefit in a versor description of the applications, for instance in virology, lies in the simple construction and implementation of the chiral and full polyhedral groups. The Clifford approach then also yields a simple construction of the binary polyhedral groups, and in fact all three groups can be straightforwardly treated in the same framework. This seemingly unknown construction of the binary polyhedral groups also sheds light on the fact why they generate Coxeter groups of rank 4. The natural 4D Euclidean structure of the spinors allows for an alternative interpretation as vectors (in fact, a root system) in a 4D space, which generate Coxeter groups in four dimensions. Thus, one can construct many four-dimensional (exceptional) Lie and Coxeter groups from three-dimensional considerations alone.

We are currently applying the more formal considerations of our recent work to extending the existing paradigm for modeling virus and fullerene structure [12] and to packing problems [27]. The chiral and binary polyhedral groups are attractive as discrete symmetry groups for flavour and neutrino model building in particle

physics, and we are currently working on an anomaly analysis (breaking of classical symmetries by quantum effects) for these groups [13].

Acknowledgements I would like to thank my family and friends for their support, my former PhD supervisor Anthony Lasenby for getting me interested in and teaching me GA (amongst many other things), as well as my collaborators Joan Lasenby, Eckhard Hitzer, Reidun Twarock, Mike Hobson, Céline Böhm, Christoph Luhn and Silvia Pascoli for helpful discussions.

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